CONVECTIVE STABILITY OF A LAYER OF MAGNETIZABLE LIQUID WITH SOLID BOUNDARIES

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The convective stability of a layer of magnetizable liquid with solid boundaries in inhomogeneous transverse and longitudinal magnetic fields is studied. Magnetic field distortion produced by the nonisothermal state of the liquid is considered.

Convective stability of a magnetizable liquid is usually studied with the assumption that the magnetic field is specified [1-5]. However, temperature perturbations lead to changes in magnetic field intensity, which, as was shown in [6, 7], can produce a significant effect on the liquid's convective stability. Thus, in [7] convective stability of a horizontal layer of magnetizable liquid in a longitudinal homogeneous magnetic field was studied with consideration of distortions (perturbations) of the field produced by temperature perturbations. It was shown that in this case magnetic field perturbations lead to stabilization of the layer relative to perturbations, the wave vector of which is directed along the field. In a study of stability of a horizontal layer of magnetizable liquid heated from below in a transverse homogeneous magnetic field [6], although field perturbations were considered, their role was not clarified. Thus, the results obtained in those studies indicate that the effect of magnetic field perturbations on convective stability of a magnetizable liquid may depend on the direction of the equilibrium magnetic field, but this dependence has not yet been clarified.

In order to determine the manner in which the direction of the equilibrium magnetic field affects the contribution of magnetic field perturbations to the thermomagnetic instability mechanism, we will consider the convective stability of a plane layer of magnetizable liquid with solid boundaries, heated from below, in a transverse $H_z(z)$ and longitudinal $H_x(z)$ magnetic field with a constant gradient along the z axis (the z axis of the Cartesian coordinate system will be directed perpendicular to the layer, and the x and y axes, along the layer). These two problems differ only in the direction of the magnetic field. We assume that gravitational forces are absent and that liquid magnetization is described by a linear "magnetic state" equation

$$M = M^* + \chi (H - H^*) - (K + \beta M^*)\,\Theta. \tag{1}$$

We note that the magnetic fields $H_x(z)$ and $H_z(z)$ are not exact solutions of Maxwell's equations. However, it can be shown that this formulation is a limiting case of the rigidly formulated problem of stability of a cylindrical layer, stated in the following manner. A magnetizable electrically nonconductive incompressible liquid is located in the gap between two solid cylindrical surfaces, the temperature of which is specified (the inner cylinder temperature T_1 is higher than the outer cylinder temperature T_2). It is assumed that the roles of internal rotations, magnetostriction, and the magnetocaloric effect are insignificant and that gravitational forces are absent. In this case a mechanical equilibrium exists within the magnetizable liquid:

$$\vec{v}_0 = 0; \quad T_0 = \frac{T_1 \ln (r/r_2) - T_2 \ln (r/r_1)}{\ln (r_1/r_2)}.$$

In this case Maxwell's equations permit equilibrium one-direction solutions: in one of the cases the magnetic field has only an azimuthal component $H = H_\varphi(r) = I/2\pi r$ and can be created by a current I flowing through the inner cylinder, while in the second case there is only a radial component; i.e.,

$$H = H_r(r) = \frac{\text{const}}{r} + \frac{(K + \beta M^*)\,[T_2 \ln (r/r_1) - T_1 \ln (r/r_2)]}{(1+\chi)\ln (r_2/r_1)}.$$

It is assumed here that the cylindrical layer of magnetizable liquid is surrounded by a nonmagnetic mass and that the gap between the cylinders is small, i.e., $l/R_0 \ll 1$ [$l = r_2 - r_1$; $R_0 = (r_1 + r_2)/2$]. We expand the expressions obtained for T_0, $H_\varphi(r)$, $H_r(r)$ in powers of l/R_0 and z/R_0, where $z = r - R_0$. Limiting ourselves to the first terms of these expansions, we obtain

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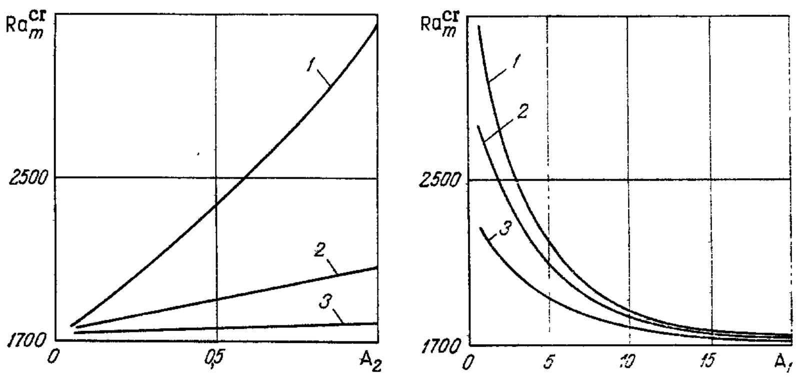

Fig. 1 Fig. 2

Fig. 1. Function $Ra_m^{cr}(A_2)$ for various values of parameter A_1: 1) A_1 = 1; 2) 5; 3) 15.

Fig. 2. Function $Ra_m^{cr}(A_1)$ for various values of parameter A_2: 1) A_2 = 1; 2) 0.5; 3) 0.3.

$$T_0 = -(T_1 - T_2)\, z/l + (T_1 + T_2)/2,$$
$$H_\varphi(r) = H_x(z) = (I/2\pi R_0)\,(1 - z/R_0),$$
$$H_r(r) = H_z(z) = \mathrm{const}\,(1 - z/R_0)/R_0 + (K + \beta M^*)\,(T_1 - T_2)\, z/(1 + \chi)\, l.$$

Thus, the study of convective stability of a thin cylindrical layer of magnetizable liquid in magnetic fields $H_\varphi(r)$ and $H_r(r)$ is mathematically equivalent to study of convective stability of a plane layer of magnetizable liquid in longitudinal and transverse magnetic fields with a constant transverse gradient. We will consider each of these cases separately.

1. $\vec{H} = [H_x(z) = H^* + Gz;\ 0;\ 0]$. We will assume that perturbations of velocity, temperature, and magnetic field satisfy a system of linear ferrohydrodynamics equations in the Boussinesq approximation. We choose as scale coefficients the thickness of the layer l for the coordinate, l^2/ν for time, ν/l for velocity, γl for temperature, Gl for magnetic field, and eliminate gradient terms from the motion equations. Then in the weak field inhomogeneity approximation $H^* \gg Gl$ the system of defining dimensionless equations will have the form [10]

$$\frac{\partial}{\partial t}\,\Delta v_z = \Delta^2 v_z + \mathrm{Gr}_m\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Theta - \mathrm{Gr}_m\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \Phi}{\partial x_j}, \tag{2}$$

$$\frac{\partial \Theta}{\partial t} = v_z + \frac{1}{\mathrm{Pr}}\,\Delta\Theta, \tag{3}$$

$$A_1\,\Delta\Phi + (1 - A_1)\,\frac{\partial^2 \Phi}{\partial x_j^2} - A_2\,\frac{\partial \Theta}{\partial x_j} = 0. \tag{4}$$

Here $\partial/\partial x_j$ is the derivative along the Cartesian coordinate along which the equilibrium magnetic field is directed. The longitudinal field case considered here corresponds to $x_j = x$; $\mathrm{Gr}_m = \mu_0(K + \beta M^*)\gamma G l^4/\rho^* \nu^2$ is the Grashof magnetic number. The criterion $A_1 = (1 + M^*/H^*)/(1 + \chi)$ characterizes the deviation of the "magnetic state" equation from linearity, while $A_2 = (K + \beta M^*)\gamma/(1 + \chi)G$ characterizes the ratio of the inhomogeneity in the magnetic field produced by the nonisothermal state of the liquid to the value of the characteristic magnetic field gradient.

Introducing normal perturbations

$$v_z,\ \Theta,\ \Phi \sim \exp i\,(k_x x + k_y y), \tag{5}$$

we obtain from system (2)-(4) equations for the amplitudes:

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 v - \mathrm{Ra}_m\, k^2\Theta + i\,\mathrm{Ra}_m\, k^2 k_x \Phi = 0, \tag{6}$$

$$v + \left(\frac{d^2}{dz^2} - k^2\right)\Theta = 0, \tag{7}$$

$$-A_2\, i k_x \Theta - (k_x^2 + A_1 k_y^2)\,\Phi + A_1\,\frac{d^2\Phi}{dz^2} = 0. \tag{8}$$

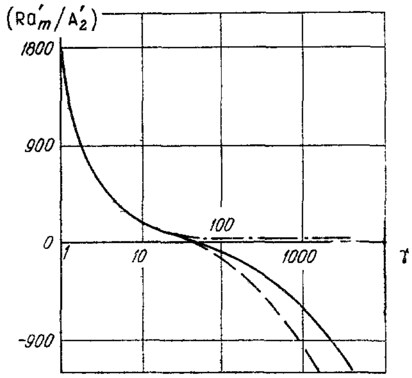

Fig. 3. Stability curves $\gamma \sim (Ra'_m A'_2)^{1/2}$, ordinate axis $(Ra'_m/A'_2)^{1/2}$.

If the layer is bounded by rigid walls, the temperature of which is specified, then the boundary conditions for velocity and temperature have the form

$$v = \frac{dv}{dz} = \Theta = 0 \quad \text{for } z = \pm 1/2. \tag{9}$$

The boundary conditions for the potential Φ are complicated by the fact that periodic magnetic field perturbations produced within the liquid by temperature perturbations induced a periodic magnetic field outside the layer. If the layer of magnetizable liquid is bounded by nonmagnetic semispaces, then outside the layer the magnetic potential ψ will be defined by the equation

$$\left(\frac{d^2}{dz^2} - k^2\right) \Psi = 0. \tag{10}$$

From the condition of continuity of the normal component of induction and the tangential component of magnetic field intensity on the boundary, it follows that

$$\Psi = \Phi; \quad \frac{d\Psi}{dz} = (1 + M^*/H^*) \frac{d\Phi}{dz}. \tag{11}$$

Substituting the solutions of Eq. (10) in Eq. (11) gives boundary conditions for Φ, analogous to those obtained in [6]:

$$(1 + M^*/H^*) \frac{d\Phi}{dz} \pm k\Phi = 0 \quad \text{for} \quad z = \pm 1/2. \tag{12}$$

System (6)-(8) is invariant for the transformation $z \to -z$, so that the exact solutions for perturbations of velocity, temperature, and magnetic field are even. We will solve the system of equations (6)-(8) with boundary conditions Eqs. (9), (12) by the Galerkin method, writing the velocity and temperature in the form

$$v = \sum_{i=1}^{n} a_i (z^2 - 1/4)^{i+1}; \quad \Theta = \sum_{i=1}^{n} b_i (z^2 - 1/4)^i. \tag{13a}$$

The functions for the velocity and temperature perturbations satisfy the boundary conditions.

Use of the Galerkin method with equations of the form of Eq. (8) is somewhat specialized because of the form of the boundary conditions for the magnetic field perturbation potential: In these cases it is sufficient to have a system of functions which satisfy only the completeness condition [8, 9]. Therefore, the function for the magnetic field perturbation potential can be chosen for convenience in further integration in the most simple form:

$$\Phi = \sum_{i=1}^{n+2} c_i z^{2(i-1)}. \tag{13b}$$

Following the Galerkin method, we require that Eq. (6) be orthogonal to every test function for velocity, while Eq. (7) must be orthogonal to every temperature test function. For Eq. (8), due to the fact that test functions for the magnetic field perturbation potential Eq. (13b) do not satisfy the boundary conditions Eq. (12), orthogonality is understood in the following sense [9, 10]:

$$\int_{-1/2}^{+1/2} \Phi_j \left[A_1 \frac{d^2\Phi}{dz^2} - (k_x^2 + A_1 k_y^2)\, \Phi - i\, A_2\, k_x \Theta \right] dz - \Phi_j\,(1/2) \left[\frac{d\Phi}{dz} \cdot \right.$$

$$\left. + \frac{k}{1 + M^*/H^*}\, \Phi \right]_{z=1/2} + \Phi_j(-1/2) \left[\frac{d\Phi}{dz} - \frac{k\Phi}{1 + M^*/H^*} \right]_{z=-1/2} = 0. \tag{14}$$

As a result, we obtain from the orthogonality conditions a system of homogeneous linear equations for the coefficients of the expansions of the test functions. This system of equations has a solution when and only when its determinant is equal to zero, which leads to a characteristic equation for determining the eigenvalue Ra_m, which in the first approximation of the Galerkin method may be found analytically:

$$Ra_m = 19600\,(0.8 + 4k^2/105 + k^4/630)\,(1/3 + k^2/30)/k^2 + Ra_m\, A_2 k_x^2\, (14\Delta_{31} - \Delta_{32}/6 + \Delta_{33}/72)/\Delta_3, \tag{15}$$

where Δ_3, Δ_{31}, Δ_{32}, Δ_{33} are third-order determinants, which will not be presented here because of space limitations.

From analysis of Eq. (15) it follows that the parameter Ra_m is minimal at $k_x = 0$ (critical motions are swells with axes parallel to the equilibrium magnetic field) and that its critical value is 1750. There is an analogy here with gravitation. We note that for stability of a layer of nonmagnetic liquid with rigid walls in a gravitational field, the Galerkin method with test functions (13) in the first approximation gives a critical Rayleigh number of 1750; in the second approximation, 1708.8, and in the third, 1707.77; i.e., an accuracy of ~2% is produced even in the first approximation [6]. The change in the structure of critical motions produced by a longitudinal magnetic field can easily be observed in experiment even in weak magnetic fields.

2. $\vec{H} = [0;\, 0;\, H_z(z) = H^* + Gz]$. We will consider the stability of a layer of magnetizable liquid heated from below in a transverse inhomogeneous magnetic field $H_z(z)$. Perturbations of velocity, temperature, and magnetic field are described by the system of equations (2)-(4), $x_j = z$. Considering, as before, normal perturbations, we obtain

$$\begin{gathered} \left(\frac{d^2}{dz^2} - k^2 \right)^2 v - Ra_m k^2 \left(\Theta - \frac{d\Phi}{dz} \right) = 0; \\ v + \left(\frac{d^2}{dz^2} - k^2 \right) \Theta = 0; \\ \frac{d^2\Phi}{dz^2} - A_1 k^2 \Phi - A_2 \frac{d\Theta}{dz} = 0. \end{gathered} \tag{16}$$

Velocity and temperature perturbations on the boundaries satisfy Eq. (9). For the magnetic field perturbation potential Φ the boundary conditions are obtained in the same manner as in the first case, but have a somewhat different form [6]:

$$\frac{d\Phi}{dz} \pm \frac{k}{1 + \chi}\, \Phi = 0. \tag{17}$$

The system of equations (16) is invariant for the transformation $z \to -z$, $\Phi \to -\Phi$, so that exact solutions for velocity and temperature perturbations are even, while those for magnetic field perturbations are odd.

We will solve system (16) with boundary conditions (9), (17) by a variant of the Galerkin method described above, representing the velocity, temperature, and magnetic field in the form of power series

$$v = \sum_{i=1}^{n} a_i\,(z^2 - 1/4)^{i+1}; \quad \Theta = \sum_{i=1}^{n} b_i\,(z^2 - 1/4)^i; \quad \Phi = \sum_{i=1}^{n+2} c_i z^{2i-1}.$$

In the first approximation ($n = 1$) the problem is solvable analytically and leads to the following equation for the stability boundary:

$$Ra_m = 19600\,(0.8 + 4k^2/105 + k^4/630)\,(1/3 + k^2/30)/k^2 + Ra_m\, A_2\,(14 d_{13}/3 - d_{23}/2 + 5 d_{33}/72)/d_3, \tag{18}$$

where d_3 is a third-order determinant with elements

$$\begin{gathered} D_{12} = D_{21} = A_1 k^2/80 + k/8\,(1 + \chi) + 1/4; \\ D_{13} = D_{31} = D_{22} = A_1 k^2/448 + k/32\,(1 + \chi) + 1/16; \\ D_{23} = D_{32} = A_1 k^2/2304 + k/128\,(1 + \chi) + 15/7 \cdot 2^6; \\ D_{11} = A_1 k^2/12 + k/2\,(1 + \chi) + 1; \;\; D_{33} = A_1 k^2/11 \cdot 2^{10} + k/2^9\,(1 + \chi) + 25/9 \cdot 2^8, \end{gathered}$$

and d_{i3} are obtained by replacement of the i-th column of determinant d_3 by the column (1/6, 1/40, 1/224). From Eq. (18) it follows that magnetic field perturbations involve the beginning of convective instability, increasing the critical values of Ra_m; i.e., in this case, generally speaking, there is no analogy with gravitation. But if the magnetic field distortions produced by temperature perturbations are small, then the magnetic field can be considered specified.

The general condition for neglect of magnetic field perturbations was obtained in [10] by the method of similarity theory

$$A_2 \ll A_1. \tag{19}$$

To produce a concrete meaning for condition (19), we will analyze the dependence of Ra_m^{cr} on the parameter A_2, shown in Fig. 1, for various values of the parameter A_1.

It is evident from the figure that for linear dependence of liquid magnetization on magnetic field intensity ($A_1 = 1$), the contribution of magnetic field perturbations to critical magnetic Rayleigh number Ra_m^{cr} will be less than 10% for $A_2 < 0.25$, while the value of this contribution decreases with growth of A_1.

For a magnetizable liquid with parameters $\gamma \sim 10^2$ deg/m, $M^* \sim 10^3$ A/m, $\beta \sim 10^{-3}$ 1/deg, $K \sim 10^3$ A/m · deg the contribution of magnetic field perturbations to the critical temperature gradient will be less than 10% in fields with a gradient $G > 4 \cdot 10^5$ A/m^2.

In order to clarify the effect on convective stability of the relationship between field and magnetization, Fig. 2 shows the critical value of Rayleigh number as a function of the nonlinearity parameter A_1 for various values of the dimensionless complex A_2. It is evident that at $A_1 \gg A_2$, Ra_m^{cr} tends to 1750, i.e., the gravitation analogy exists, or, in other words, at $A_1 \gg A_2$ magnetic field perturbations have no significant effect on convective stability of a magnetizable liquid [at $A_1 > 7$ in a homogeneous magnetic field ($A_2 = 1$) the contribution of magnetic field perturbations to the critical magnetic Rayleigh number values is less than 10%]. For available magnetic liquids $A_1 \sim 1$, but in a purely formal manner we can consider variation of this parameter over wide bounds.

The dimensionless parameters Ra_m and A_2 contain the value of the characteristic magnetic field gradient in a nonisothermal magnetizable liquid G. We note that G does not always coincide with the characteristic magnetic field gradient in an isothermal liquid G' (we will term the latter the external gradient), and in the problem under consideration includes still another magnetic field gradient produced by the temperature difference across the layer boundaries; i.e.,

$$G = G' + (K + \beta M^*)\,\gamma/(1+\chi).$$

In order to clarify the dependence of critical external magnetic field gradient value on temperature gradient, we compose dimensionless complexes, one of which

$$(Ra_m' A_2')^{1/2} = (K + \beta M^*)\,\gamma l^2\,[\mu_0/\nu\varkappa\rho^*(1+\chi)]^{1/2}$$

does not contain the magnetic field gradient, while the second

$$(Ra_m'/A_2')^{1/2} = G' l^2\,[\mu_0(1+\chi)/\nu\varkappa\rho^*]^{1/2}$$

does not contain the temperature gradient; we use here the notation

$$Ra_m' = \mu_0 (K + \beta M^*)\,G' \gamma l^4/\rho^* \nu \varkappa, \quad A_2' = (K + \beta M^*)\,\gamma/(1+\chi)\,G'.$$

Then the equation for the stability limit leads to the following dependence of dimensionless external magnetic field gradient $(Ra_m'/A_2')^{1/2}$ on dimensionless temperature gradient $(Ra_m' A_2')^{1/2}$:

$$(Ra_m'/A_2')^{1/2} = 19600\,(0.8 + 4k^2/105 + k^4/810)/k^2 (Ra_m' A_2')^{1/2} + (Ra_m' A_2')^{1/2}\,(14d_{13}/3 - d_{23}/2 + 5d_{33}/72 - d_3)/d_3.$$

The stability limit curve (Fig. 3) is shown on the coordinate plane $\{(Ra_m'/A_2')^{1/2}, (Ra_m' A_2')^{1/2}\}$ by a solid line [the dimensionless gradient $(Ra_m' A_2')^{1/2}$ is depicted in logarithmic scale in Fig. 3]. The dashed line shows the stability curve which would be obtained without consideration of magnetic field perturbations. If, in studying the instability of a horizontal layer of magnetizable liquid in an inhomogeneous magnetic field $H_z(z)$ we assume that the magnetic field is specified, i.e., if for the characteristic magnetic field gradient G we choose the external gradient G', then the stability limit curve will be a hyperbola (dash – dot line of Fig. 3). It is evident from Fig. 3 that such an approach, i.e., the magnetic field specified and set equal to the field in an isothermal magnetizable liquid, is applicable for temperature gradients $(Ra_m' A_2')^{1/2} < 50$.

Thus, we can conclude that the character of the effect of magnetic field perturbations on convective stability of a layer of magnetizable liquid is determined by both the intensity gradient and the direction of the equilibrium magnetic field. In other words, the critical Rayleigh number value corresponds to convective motions not produced by magnetic field perturbations along the field axis. If such motions are impossible in a given geometry, then the magnetic field perturbations which do develop are involved with the beginning of convective instability, increasing the values of the critical parameters.

NOTATION

$\vec{M}$, $\vec{H}$, liquid magnetization and magnetic field strength vectors; M, H, magnitudes of these vectors; T, liquid temperature; ρ, density; *, index denoting mean values over layer; χ, magnetic susceptibility; ν, coefficient of kinematic viscosity; $\varkappa$, thermal diffusivity; γ, characteristic temperature gradient; G, characteristic magnetic field gradient; K, pyromagnetic coefficient; $\beta = (-1/\rho)(\partial\rho/\partial T)$; r_1, r_2, radii of internal and external boundaries of cylindrical layer; v, Θ, dimensionless perturbations; z, velocity and temperature components; Φ, dimensionless magnetic field perturbation potential ($H_1 = \nabla\Phi$); $k^2 = k_x^2 + k_y^2$, square of dimensionless wave number; $Pr = \nu/\varkappa$, dimensionless Prandtl number.

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